$\mathbb{R}$ is a complete ordered field with $+$, $\times$, and $\leq$

i.e. it satisfies the following three sets of axioms: I, II & III

I. Field axiom:

(1) $\exists\ 0 \in \mathbb{R}$ such that $x + 0 = x = 0 + x \ \forall x \in \mathbb{R}$;

(2) $\forall x \in \mathbb{R}, \exists -x \in \mathbb{R}$ s.t. $x + (-x) = 0 \ (= (-x) + x)$;

(3) $(x+y)+z = x + (y+z) \ \forall x, y, z \in \mathbb{R}$;

(4) $(x+y) = y+x \quad \forall x, y \in \mathbb{R}$;

(5) $\exists\ 1 \in \mathbb{R}$ such that $x \cdot 1 = x = 1 \cdot x \quad \forall x \in \mathbb{R}$;
(in particular $0 \cdot 1 = 0$)

(6) $\forall x \in \mathbb{R}$ with $x \neq 0$, $\exists$ ~~$x \neq 0$~~ such that (s.t.) $x x' = 1 = x' x$

(7) $(xy)z = x(yz), \ \forall x, y, z \in \mathbb{R}$;

(8) $xy = yx \ \forall x, y \in \mathbb{R}$;

(9) $x(y+z) = xy + xz$ & $(y+z)x = \cdots \quad \forall x, y, z \in$

(10) $1 \neq 0$. (so $\mathbb{R}$ has at ~~least two~~ elements)

Remarks/Exercises

(i) In (1), (2), (5), (6), $\exists$ can be replaced by $\exists!$ (exists uniquely)

e.g. if also $x + 0' = x = 0' + x \ \forall x \in \mathbb{R}$ then (why)

$$0 + 0' = 0 \quad \& \quad 0 + 0' = 0' \quad (\text{so } 0 = 0')$$

(ii) Show that $(-1) x = -x$ $\left(\because (-1)x + x = ((-1)+1)x = 0\right.$
and $(-1)(-1) = 1$.

(iii) Show the "cancellation laws" and other familiar formulae learnt in ~~school~~ (eg $(a+b)^2 = a^2 + 2ab + b^2$)

(iv) $X = \{0\}$ satisfies (1) – (9)
$X = \mathbb{N}/\text{mod } 2 = \{0, 1\}$ satisfies (1) – (10) of (I).
(so eg $1+1=0$)

II. Order-field axiom (with $\leq$; $<$ is meant ($x \leq y$ & $x \neq$

1) $\leq$ is transitive, i.e. $x \leq y$ and $y \leq z$ imply $x \leq z$.
2) trichotomy, i.e. exactly one of $x < y$ or $x = y$ or $y < x$ (for any $x, y \in \mathbb{R}$).

EX2 $x \leq y$ and $y \leq x$ imply $x = y$.

3) $\leq$ is compatible w.r.t $+$ and positive-multiplication, i.e.,
if $x_i \leq y_i$ $(i = 1, 2)$ then $x_1 + x_2 \leq y_1 + y_2$;
if $x \leq y$ and $0 < \lambda$ then $\lambda x \leq \lambda y$;

EX3. 1) $x \leq y$ iff $0 \leq y - x$ $(y - x := y + (-x))$
2) $x \leq y$ iff $-y \leq -x$
3) if $x \leq y$ and $\lambda < 0$ then $\lambda x \geq \lambda y$ (meaning that $\lambda y \leq \lambda x$)

Remarks/Ex. Let $\mathbb{N}$ be the smallest inductive set in $\mathbb{R}$ with $1 \in$, that is $\mathbb{N}$ is a (the) smallest subset of $\mathbb{R}$ s.t. $1 \in \mathbb{N}$ and $n+1 \in \mathbb{N}$ whenever $n \in \mathbb{N}$.

1. Show the MI (Math Induction). Let $n \in \mathbb{N}$ and $P(n)$ ….
(Hint. Let $\mathbb{N}_0 := \{1\} \cup \{n \in \mathbb{N} : P(n) \text{ holds}\}$.)

2. Let $A \subseteq \mathbb{N}$ be a finite set: $\#(A) = n$ with some $n \in \mathbb{N}$ (that A and $\{1, 2 \cdots n\}$ are in one-one correspondence). Then A has the smallest ele, and the largest ele. (induction over
(This result is known as the well-order principle).

3 Any non-empty subset of $\mathbb{N}$ has the smallest ele.
(not true for "largest"!) Let $a_0 \in A$ and
$A_0 := \{a \in A : a \leq a_0\}$. Then A_0 is finite and its smallest ele is also the smallest ele of A.

4. 1 is the smallest ele of $\mathbb{N}$ and 2 is the smallest in $\mathbb{N} \setminus \{1\}$ (Hint, consider $\mathbb{N}_0 = \{1\} \cup \{n \in \mathbb{N} : 2 \leq n\}$

Consequences of I & II.

(i) $\mathbb{N}$ is, by def, the smallest inductive subset of $\mathbb{R}$

$1 \in \mathbb{N}$ &

$n+1 \in \mathbb{N}$ whenever $n \in \mathbb{N}$.

(ii) $0 < 1 < 2 < 3 < \cdots$ (so $\mathbb{N}$ contains infinitely many)

$\nexists\, n \in \mathbb{N}$ s.t. $0 < n < 1$ (as $\{n \in \mathbb{N} : 1 \le n\}$ is inductive)

(iii) $\mathbb{Z} := \mathbb{N} \cup \{0\} \cup \{-n : n \in \mathbb{N}\}$ is also inductive

(iv) Since $1 \cup \{n \in \mathbb{N} : 2 \le n\}$ is inductive, 2 is the second smallest in $\mathbb{N}$

Th1 (M.I). (can start from 1 or any $m \in \mathbb{Z}$).

Th2 (well-order principle).

(i) Any nonempty finite set of real nos. has the smallest ele.

" " largest ele.

(ii) Any nonempty (finite or infinite) subset of $\mathbb{N}$ has the smallest ele. (not true if not "of $\mathbb{N}$")

III. Order-completeness axiom.

If $A \subseteq \mathbb{R}$ has an upper bound (u.b.) then $\exists$ a (the) least u.b. of A, namely if

$\exists\ u \in \mathbb{R}$ s.t. $a \leq u\ \forall a \in A$ then

$\exists\ \bar{u} \in \mathbb{R}$ s.t. a) $a \leq \bar{u}\ \forall a \in A$ &

b) $\bar{u} \leq u$ whenever $a \leq u\ \forall a \in A$.

where b) can be equivalently stated as

b′) if $v < \bar{u}$ then $v < a$ for some $a \in A$.

Th. If $B \subseteq \mathbb{R}$ has a l.b. (in the sense that $\exists\ \ell \in \mathbb{R}$ s.t. $\ell \leq b\ \forall b \in B$) then

$\exists\ \bar{\ell} \in \mathbb{R}$ s.t. α) $\bar{\ell} \leq b\ \forall b \in B$

β) $\ell \leq \bar{\ell}$ whenever $\ell \leq b\ \forall b \in B$

Pf. Let $A := \{-b : b \in B\}$ and let ℓ be a l.b. of B: $\ell \leq b\ \forall b \in B$. Then $-b \leq -\ell\ \forall b \in B$, showing that A has an u.b (−ℓ). Hence, by III, $\exists\ \bar{u}$ satisfying a) & b) mentioned above.

By a) $-\bar{u} \leq -a\ \forall a \in A$, i.e. $-\bar{u} \leq b\ \forall b \in B$ showing that $\bar{\ell} := -\bar{u}$ has the property α). To check β), suppose $\ell \leq b\ \forall b \in B$ (has to show that $\ell \leq \bar{\ell}$)

Notice that $-b \leq -\ell\ \forall b \in B$ and so

$$a \leq -\ell\ \forall a \in A$$

By b) above, $\bar{u} \leq -\ell$ and so $\ell \leq -\bar{u} = \bar{\ell}$ as required ~~to~~ show.

Consequence of Completeness

Th (Archimedean Property). For any $r \in \mathbb{R}$, $\exists\, n \in \mathbb{N}$ s.t. $r < n$.

Pf. Otherwise, let $s := \sup \mathbb{N}$ (so in $\mathbb{R}$, why?). Then $\exists\, m \in \mathbb{N}$ s.t. $s - 1 < m$ and so $n \leq s < s+1 < m+1$ $\forall n \in \mathbb{N}$. Letting $n := m+1$, one has $m+1 < m+1$, which is absurd (contradicting to what?)

Cor 1. Let $\emptyset \neq A \subseteq \mathbb{N}$. Then $\circlearrowleft$ (the following statements are equi.)

(i) A is bounded above: $\exists\, u \in \mathbb{R}$ s.t. u is an u.b. of A;

(ii) $\exists\, n \in \mathbb{N}$ s.t. n is an u.b of A;

(iii) A is a finite set.

Cor 2. Let $\emptyset \neq A \subseteq \mathbb{R}$ be bounded below. Then $\exists\, m \in \mathbb{N}$ such that A is bounded below by $-m$.

Consequently if $\emptyset \neq A \subseteq \mathbb{Z}$ (integers) bounded below then A has the smallest ele. (Hint: $\{m + a : a \in A\}$ has the smallest ele.)

Ex. Establish the corr. result for bounded above nonempty subset A of $\mathbb{Z}$.

Cor 3 Let r be a positive real number. Then $\exists\, n \in \mathbb{N}$ s.t. $\frac{1}{n} < r$.

Cor 4. (Integral Part) Let $x \in \mathbb{R}$. Then $\exists\, \bar{m} \in \mathbb{Z}$ s.t. $\bar{m} \leq x < \bar{m}+1$ (i.e. $[x] = \bar{m}$).

Proof. Let $M := \{m \in \mathbb{Z} : m \leq x\}$. Then $M \neq \emptyset$ and bounded above (why??) so $\exists\, \bar{m} := \max M$ (the

Cor 5. (Density of $\mathbb{Q}$). Let $x, y \in \mathbb{R}$ with $x < y$. Then $\exists\ z \in \mathbb{Q}$ s.t. $x < z < y$.

Pf. Suppose in addition that $1 < y - x$. Then take $\bar{m} = [x]$ and so $z := \bar{m} + 1$ has the desired property

$$\bar{m} \le x < \bar{m} + 1 \quad (\le x + 1 < y).$$

General case of $x < y$. Then take $n \in \mathbb{N}$ s.t. $\frac{1}{n} < y - x$ (so $1 < (ny) - (nx)$). By the first part, $\exists\ z \in \mathbb{Z}$ s.t. $(nx) < z < (ny)$ (i.e. $x < \frac{z}{n} < y$ & $\frac{z}{n} \in \mathbb{Q}$).

2nd method. Take $n \in \mathbb{N}$ and $m \in \mathbb{N}$ s.t. $\frac{1}{n} < y - x$ and $-m < x$. Let

$$A := \{ k \in \mathbb{N} \cup \{0\} : -m + \frac{1}{n} \cdot k \le x \}$$

and let $\bar{k} = \max A$ (why exists?) and $z := -m + \frac{1}{n}(\bar{k} + 1)$; then $z \in \mathbb{Q}$.

$$x < z \le x + \frac{1}{n} \cdot 1 < y.$$

Cor 6. Existence of $\sqrt{2}$ (Ex. show that $\sqrt{2} \notin \mathbb{Q}$).

Pf. Let $S = \{ m \in \mathbb{N} : m^2 < 2 \}$ ($\neq \emptyset$, e.g., $1 \in S$). If $m \in S$ then $m^2 < 2 < 2^2$ and so $0 < (2 - m)(m + 2)$ implying that $m < 2$ (so S is bounded by 2). Let $x \in \sup S$ (≤ 2); then $1 \le x$. Will show that $x^2 = 2$ by rejecting the following two cases

(i) Suppose $x^2 < 2$. Then $\exists\, m \in \mathbb{N}$ such that

$$\left(x+\frac{1}{m}\right)^2 = x^2 + \frac{2x}{m} + \frac{1}{m^2} \leq x^2 + \frac{1}{m}(2x+1) < 2$$

(why such m exists?), showing that $x + \frac{1}{m} \in S$ contradicting that $x \in \sup S$.

(ii) Suppose $2 < x^2$. Then $\exists\, m \in \mathbb{N}$ s.t.

$$\left(x-\frac{1}{m}\right)^2 = x^2 - \frac{2x}{m} + \frac{1}{m^2} > x^2 - \frac{2x}{m} > 2$$

(why such m exists for the last inequality?).

You are asked to check that $x - \frac{1}{m}$ is then an u.b of S: $\forall\, k \in S$ $(k^2 < 2)$ one has

$$\left(x-\frac{1}{m}\right)^2 > 2 > k^2,$$ implying that $x - \frac{1}{m} > k$.

By the "smallest property" of $\sup S$, this is impossible.

Cor 7 $\mathbb{R} \setminus \mathbb{Q}$ also dense in $\mathbb{R}$.

Th 1. Characterization of intervals. Let $A \subseteq \mathbb{R}$ contain more than one point and be (order-) convex in the sense that $x \in A$ whenever $a_1 \leq x \leq a_2$ for some $a_1, a_2 \in A$.

Th 2. (Nested Interval Th). Let $\{I_n : n \in \mathbb{N}\}$ be a sequence of bounded closed nested intervals in the sense that $\emptyset \neq I_n \supseteq I_{n+1}$ $\forall\, n \in \mathbb{N}$. Then $\bigcap I_n \neq \emptyset$

Th (Characterization of intervals). Let $A \subseteq \mathbb{R}$ contain more than one elements and be order-convex in the sense that

(#) $x \in A$ whenever $a_1 < x < a_2$ for some $a_1, a_2 \in A$

(note: the inequality may be replaced by $a_1 \leq x \leq a_2$; why?)

Then A is an interval.

Pf. Consider first the case that A is bounded (above and below). Then, by completeness of $\mathbb{R}$, let

$$\alpha := \inf A \quad \& \quad \beta = \sup A \quad (\alpha, \beta \in \mathbb{R}).$$

Then $\alpha \leq \beta$ (in fact $\alpha < \beta$ as A contains more than one point say $a', a'' \in A$ with $a' < a''$. Then $\alpha \leq a' < a'' \leq \beta$ so $\alpha < \beta$).

We will show that

$$(*) \quad (\alpha, \beta) \subseteq A \subseteq [\alpha, \beta],$$

(where the 2nd inclusion by the definitions of α, β).

To see the 1st inclusion, let $x \in (\alpha, \beta)$. Then $x < \beta$ and so x is not an upper bound of A (because β is the least upper bound of A), and so $x < a_2$ for some $a_2 \in A$. Similarly, since $\alpha < x$, there exists some $a_1 \in A$ such that $a_1 < x$. By (#) and the facts that $a_1 < x < a_2$ and $a_1, a_2 \in A$, one concludes that $x \in A$. Thus (*) is proved. This entails that A is an interval (with end points α, β). Indeed, suppose $(\alpha, \beta) \neq A \neq [\alpha, \beta]$. Take $\gamma \in A \setminus (\alpha, \beta)$.

Then $\gamma \in [\alpha,\beta] \setminus (\alpha,\beta) = \{\alpha,\beta\}$. If $\gamma = \alpha$ then $A = [\alpha,\beta)$, while if $\gamma = \beta$ then $A = (\alpha,\beta]$.

Next consider the case (iv) when A is neither bounded below nor bounded above. Then $A \supseteq \mathbb{R}$ (so equal as $A \subseteq \mathbb{R}$). Indeed if $x \in \mathbb{R}$ then x is neither an upper bound nor a lower bound of A so there exist $\bar{a}, \bar{\bar{a}} \in A$ such that

$$\bar{a} < x < \bar{\bar{a}},$$

which implies that $x \in A$ by (#), as claimed.

Finally, the remaining cases (namely
(ii) A is bounded above but not below
(iii) A is bounded below but not above)
can be dealt with similarly. We leave the verifications ~~to~~ the reader (as "homeworks").

Nested Interval Th. Let $I_n = [a_n, b_n] \subseteq \mathbb{R}$ $\forall n \in \mathbb{N}$ with $a_n \le b_n$, and suppose that

$$I_n \supseteq I_{n+1} \quad \forall n \in \mathbb{N}.$$

Then $\bigcap_{n\in\mathbb{N}} I_n \neq \emptyset$.

Let $n \le m$ (in $\mathbb{N}$). Then $I_n \supseteq I_m \ni a_m, b_m$ so $a_n \le a_m \le b_m \le b_n$; thus $a_n, a_m \le b_m, b_n$.

This means that any member in $\{a_n : n \in \mathbb{N}\}$ is bounded above by any member in $\{b_m : m \in \mathbb{N}\}$. Consequently $\alpha = \sup\{a_n : n \in \mathbb{N}\}$ and $\beta = \inf\{b_m : m \in \mathbb{N}\}$ exist in $\mathbb{R}$ such that $\alpha \leq \beta$. Complete the proof by showing that

$$\bigcap_{n \in \mathbb{N}} I_n = [\alpha, \beta].$$

Application (to show that $[0,1]$ or any non-degenerate interval cannot be represented as

(*) $[0,1] = \{a_n : n \in \mathbb{N}\}$. (non-degenerate)

Indeed suppose (*) holds. Then $\exists$ a closed subinterval I_1 of $[0,1]$ with a positive length such that $a_1 \in [0,1] \setminus I_1$ (I_1 ~~can~~ can be chosen, say from $[0, \frac{1}{3}]$, $[\frac{1}{3}, \frac{2}{3}]$ or $[\frac{2}{3}, 1]$).

Similarly $\exists$ a closed subinterval I_2 of I_1 with pos. length such that $a_2 \in [0,1] \setminus I_2$. Inductively we have a nested sequence $\{I_n : n \in \mathbb{N}\}$ of non-degenerate bounded closed intervals such that $a_n \notin I_n \ \forall n$. Then $\exists x$ such that $x \in I_n \ \forall n$ (so $x \neq a_n \ \forall n$). Noting $x \in [0,1]$, we have a contradiction with (*).